Exercise 12

Dinosaur fossils are too old to be reliably dated using carbon-14. (See Exercise 11.) Suppose we had a 68-million-year-old dinosaur fossil. What fraction of the living dinosaur's ^{14}C would be remaining today? Suppose the minimum detectable amount is 0.1%. What is the maximum age of a fossil that we could date using ^{14}C?

Solution

Assume that the rate of mass decay is proportional to the amount of mass remaining at any given time.

$$\frac{dm}{dt} \propto -m$$

There's a minus sign here because mass is being lost as time increases. Change the proportionality to an equation by introducing a (positive) constant k.

$$\frac{dm}{dt} = -km$$

Divide both sides by m.

$$\frac{1}{m}\frac{dm}{dt} = -k$$

Rewrite the left side by using the chain rule.

$$\frac{d}{dt}\ln m = -k$$

The function you have to differentiate to get $-k$ is $-kt + C$, where C is any constant.

$$\ln m = -kt + C$$

Exponentiate both sides.

$$e^{\ln m} = e^{-kt+C}$$

$$m(t) = e^C e^{-kt}$$

Use a new constant m_0 for e^C.

$$m(t) = m_0 e^{-kt} \tag{1}$$

The half-life is defined as the amount of time it takes for a sample to decay to half its mass. Recall from the previous exercise that the half-line of carbon-14 is 5730 years, so set $m(5730) = m_0/2$ and solve the equation for k.

$$m(5730) = \frac{m_0}{2}$$

$$m_0 e^{-k(5730)} = \frac{m_0}{2}$$

$$e^{-5730k} = \frac{1}{2}$$

$$\ln e^{-5730k} = \ln \frac{1}{2}$$

$$(-5730k) \ln e = -\ln 2$$

$$k = \frac{\ln 2}{5730} \approx 0.000120968 \text{ year}^{-1}$$

As a result, equation (1) becomes

$$\begin{aligned} m(t) &= m_0 e^{-\left(\frac{\ln 2}{5730}\right)t} \\ &= m_0 e^{\ln 2^{-t/5730}} \\ &= m_0 (2)^{-t/5730}. \end{aligned}$$

The fraction of ^{14}C remaining today after 68 million years is

$$\frac{m(68 \times 10^6)}{m_0} = 2^{-(68\times 10^6)/5730} \approx \frac{1}{2^{11867.4}} \approx 0.$$

To find how long it takes for the ^{14}C to reduce to 0.1% of its original amount, set $m(t) = 0.001m_0$ and solve the equation for t.

$$m(t) = 0.001m_0$$

$$m_0(2)^{-t/5730} = 0.001m_0$$

$$2^{-t/5730} = 0.001$$

$$\ln 2^{-t/5730} = \ln 0.001$$

$$\left(-\frac{t}{5730}\right) \ln 2 = \ln 0.001$$

$$t = -\frac{5730 \ln 0.001}{\ln 2} \approx 57\,103.9 \text{ years}$$

This is the maximum age of a fossil that can be dated with ^{14}C if the minimum detectable amount is 0.1%.